These problems are designed for discussions in the classes of Week 8 of Michaelmas term.[1]

1. Private Provision of Public Good. Consider the following *public good* game:

	Contribute	Don't
Contribute	$1 - c_1, 1 - c_2$	$1 - c_1, 1$
Don't	$1, 1 - c_2$	$0, 0$

The benefits (1 to each player if at least one contributes) are commonly known, but the costs of contributing c_i is private information to player i. A *strategy* for player i in this game specifies an action ("Contribute" or "Don't") for each possible value of c_i.

(i) Suppose it is common knowledge that $c_1 = \frac{1}{4}$, but Player 1 does not know c_2. Player 1 believes $c_2 = \frac{1}{4}$ with probability $\frac{1}{2}$ and $c_2 = 2$ with probability $\frac{1}{2}$.

(a) If $c_2 = 2$, does Player 2 have a dominant strategy? If so, what is it? Does Player 2 have a dominant strategy if $c_2 = \frac{1}{4}$?

(b) Let z denote the probability that Player 2 contributes if the cost is $c_2 = \frac{1}{4}$. What is Player 1's expected payoff from "Don't", given his beliefs, in terms of z? Does Player 1 have a dominant strategy?

(c) What is the Bayesian-Nash equilibrium?

(ii) Suppose that Player i believes $Pr[c_j = \frac{1}{4}] = Pr[c_j = 2] = \frac{1}{2}$ for $i = 1, 2$ and $j = 1, 2$, $j \neq i$. What is the Bayesian-Nash equilibrium of the game?

(iii) Suppose that both players believe costs are drawn independently from a uniform distribution on the interval $[0, 2]$. What is the Bayesian-Nash Equilibrium now?[2]

2. A Bayesian Trading Game. Suppose that a buyer has a valuation for a good v_b, uniformly distributed on $[0, 1]$. The seller has a valuation v_s independently and identically distributed on $[0, 1]$. They each observe their own valuation, but not that of the other player. Simultaneously they each announce a price, p_b and p_s respectively. If $p_b \geq p_s$ a sale takes place at a price halfway between their announcements, $p = (p_b + p_s)/2$, and the buyer receives the good, yielding payoffs of

$$u_b(p, v_b) = v_b - p \quad \text{and} \quad u_s(p, v_s) = p - v_s.$$

[1]Thanks to Yuval Heller and previous lecturers for some of these questions.

[2]Hint: Show that an equilibrium strategy has the trigger form of contributing whenever $c_i \leq c_i^*$, and note the uniform distribution of costs.

Otherwise, there is no sale and both players receive 0.[3]

(i) Suppose that the seller uses a linear strategy of the form $p_s(v_s) = \alpha + \beta v_s$. Show that the buyer's expected payoff may be written

$$\frac{p_b - \alpha}{\beta}\left[v_b - \frac{1}{2}\left\{p_b + \frac{\alpha + p_b}{2}\right\}\right].$$

(ii) Hence calculate the buyer's best reply to the seller's linear strategy, and show that it is also linear.

(iii) Now suppose the buyer uses a linear strategy of the form $p_b(v_b) = \gamma + \delta v_b$. By calculating the seller's expected payoff, find the best reply of the seller to this strategy, and show that it is also linear.

(iv) Calculate values of α, β, γ and δ, such that these strategies constitute a linear Bayesian-Nash equilibrium of the trading game.

(v) For what values of v_b and v_s is trade mutually advantageous? For what values of v_b and v_s does trade take place? Comment.

(vi) Now suppose the buyer and seller use the following strategies:

$$\begin{array}{llllll} p_b = \frac{1}{2} & \text{if} & v_b \geq \frac{1}{2} & \text{and} & p_b = 0 & \text{otherwise,} \\ p_s = \frac{1}{2} & \text{if} & v_s \leq \frac{1}{2} & \text{and} & p_s = 1 & \text{otherwise,} \end{array}$$

Argue that these strategies constitute a Bayesian-Nash equilibrium of the trading game. Without doing any further calculations, are there any other Bayesian-Nash equilibria of this game? Are any of these efficient?

3. Two Stage Game. Consider the following simultaneous-move stage game:

	L	C	R
T	3, 1	0, 0	5, 0
M	2, 1	1, 2	3, 1
B	1, 2	0, 1	4, 4

This stage game is played twice, with the outcome from the first stage observed before the second stage begins. There is no discounting. Can the payoff $(4, 4)$ be achieved in the first stage in a pure-strategy subgame-perfect Nash equilibrium of the two stage game? If so, describe a strategy profile that does so and prove that it is a subgame perfect Nash equilibrium. If not, prove why not.

4. Repeated Game. Consider an infinitely repeated game where the stage game is:

[3] It would be equivalent to assume $u_s = p$ in the case of a sale and $u_s = v_s$ when there is no sale. Why?

	L	R
U	9, 9	1, 10
D	10, 1	7, 7

Players discount the future using the common discount factor δ.

(i) What outcomes in the stage-game are consistent with Nash equilibrium play?

(ii) Let v_R and v_C be the repeated game payoffs to Row and Column respectively. Draw the set of feasible payoffs from the repeated game, explaining any normalisation you use.

(iii) Are all the payoffs in the feasible set obtainable from mixed-strategy combinations in the stage-game? (That is, for every point in the feasible set, can you find a p such that $0 \leq q \leq 1$ and a q such that $0 \leq p \leq 1$ that will give those expected payoffs from a single play?)

(iv) What are the players' minmax values? Show the individually rational feasible set.

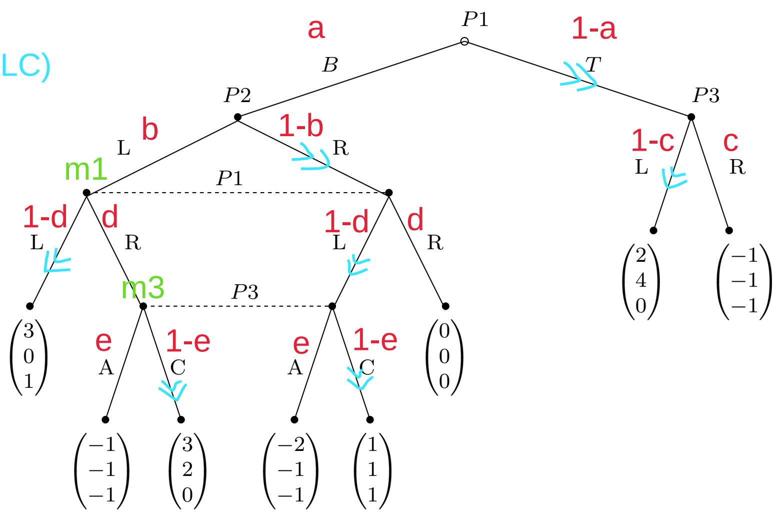
(v) Find a Nash equilibrium in which the players obtain the $(9, 9)$ payoff each period forever. What restrictions on δ are necessary?

5. Sequential Equilibrium. Consider the following extensive form game:

strategies

beliefs

(TL,R,LC)



Find the set of pure strategy Nash, subgame perfect and sequential equilibria and their payoffs. Is there any mixed sequential equilibrium of this game? If so, which ones?

6. Cournot Competition. Consider an n firm homogeneous product industry where firm i produces output q_i at cost cq_i. Price is

$$p = \alpha - \beta Q \quad \text{where} \quad Q = \sum_{i=1}^{n} q_i.$$

(i) What are the firms' outputs, prices and profits in the Cournot equilibrium? What happens as $n \longrightarrow \infty$?

(ii) Two firms merge. The merged firm has marginal costs of c, just as before. What happens to the merged firms' profits? What happens to the remaining firms' profits? Comment.

(iii) Now suppose that any firm producing positive output incurs a fixed cost of F:

$$c_i(q_i) = F + cq_i \quad \text{if} \quad q_i > 0$$

and $c_i(0) = 0$. Let $n = 4$. Suppose F satisfies:

$$F = \frac{1}{\beta}\left[\frac{2(\alpha - c)}{9}\right]^2.$$

(a) What are the pure strategy equilibria?[4] (b) Without calculations, do you think there may be any mixed equilibria?

7. An Entry Game. Consider the following two-period game with no discounting.

- In period 1, four firms simultaneously and independently decide whether or not to pay 1 to enter an industry.
- In period 2, all firms that chose to enter now simultaneously and independently choose production levels, with fixed cost $F = 5$ and zero marginal cost $c = 0$. (That is, a firm in the industry can either produce no output and incur no costs in period 2, or can produce any positive output and incur a total cost of 5 in period 2.) All production is sold at price $10 - Q$ where Q is total industry output.

Consider the five possible post-period-1 outcomes: n firms enter, where $n = 0, 1, 2, 3, 4$.

(i) Consider the Nash equilibria of the five different possible period-2 subgames corresponding to $n = 0, 1, 2, 3, 4$ entrants. (You should have a good understanding of these from the earlier question.) Which period-1 outcomes are consistent with all firms choosing pure strategies in the Nash equilibrium of the whole game that are subgame perfect (i.e., consistent with backwards induction logic)? Explain.

(ii) One of the outcomes you found in part (1) (you should have found more than one) is inconsistent with forwards induction logic. Which is it? Explain.

[4]Hint: There may be equilibria with only $m \leq n$ active firms, so try each case $m = 1, ..., 4$. Start by looking at the case where one firm is producing the monopoly output and the other firms are producing nothing. Can this be a Nash equilibrium? Does a firm nor producing in this situation have an incentive to deviate? The look at the case when two firms are producing, and so on.

(iii) In addition to the subgame-perfect outcomes, there is one outcome consistent with all firms choosing pure strategies in a Nash equilibrium of the whole game that is imperfect. Which is it? Explain.

(iv) Of the remaining period-1 outcomes, which are consistent with Nash equilibrium behaviour (perhaps including mixed strategies).